Chapter Four

Exponents and Logarithms

Very large or very small numbers or expressions can easily be expressed in writing them by exponents. As a result, calculations and solution of mathematical problems become easier. Scientific or standard form of a number is expressed by exponents. Therefore, every student should have the knowledge about the idea of exponents and its applications.

Exponents beget logarithms. Multiplication and division of numbers or expressions and exponent related calculations have become easier with the help of logarithms. Use of logarithm in scientific calculations was the only way before the practice of using the calculator and computer at present. Still the use of logarithm is important as the alternative of calculator and computer. In this chapter, exponents and logarithms have been discussed in detail.

At the end of the chapter, the students will be able to –

- Explain the rational exponent
- Explain and apply the positive integral exponents, zero and negative integral exponents
- Solve the problems by describing and applying the rules of exponents
- Explain the nth root and rational fractional exponents and express the nth root in terms of exponents
- Explain the logarithms
- Prove and apply the formulae of logarithms
- Explain the natural logarithm and common logarithm
- Explain the scientific form of numbers
- Explain the characteristic and mantissa of common logarithm and
- Find common and natural logarithm by calculator.

4.1 Exponents or Indices

In class VI, we have got the idea of exponents and in class VII, we have known the exponential rules for multiplication and division.

Expression associated with exponent and base is called exponential expression.

Activity : Fill in the blanks			
Successive multiplication of the same number or expression	**Exponential expression**	**Base**	**Power or exponent**
$2\times2\times2$	2^3	2	3
$3\times3\times3\times3$		3	
$a\times a\times a$	a^3		
$b\times b\times b\times b\times b$			5

If a is any real number, successive multiplication of n times a; that is, $a \times a \times a \times ..... \times a$ is written in the form a^n, where n is a positive integer.
$a \times a \times a \times ..... \times a$ (n times a) = a^n.

Here $n \rightarrow$ index or power
$a \rightarrow$ base

Again, conversely, a^n = $a \times a \times a \times ..... \times a$ (n times a). Exponents may not only be positive integer, it may also be negative integer or positive fraction or negative fraction. That is, for $a \in R$ (set of real numbers) and $n \in Q$ (set of rational numbers), a^n is defined. Besides, it may also be irrational exponent. But as it is out of curriculum, it has not been discussed in this chapter.

4.2 Formulae for exponents

Let, $a \in R; m, n \in N$.

Formula 1. $a^m \times a^n = a^{m+n}$

Formula 2. $\dfrac{a^m}{a^n} = \begin{cases} a^{m-n}, & \text{when } m>n \\ \dfrac{1}{a^{n-m}}, & \text{when } n>m, a \neq 0 \end{cases}$

Fill in the blanks of the following table :

a^m, a^n $a \neq 0$	$m > n$	$n > m$
	$m = 5,\ n = 3$	$m = 3,\ n = 5$
$a^m \times a^n$	$a^5 \times a^3 = (a \times a \times a \times a \times a) \times (a \times a \times a)$ $= a \times a \times a \times a \times a \times a \times a \times a$ $= a^8 = a^{5+3}$	$a^3 \times a^5 =$
$\dfrac{a^m}{a^n}$	$\dfrac{a^5}{a^3} =$	$\dfrac{a^3}{a^5} = \dfrac{a \times a \times a}{a \times a \times a \times a \times a}$ $= \dfrac{1}{a^2} = \dfrac{1}{a^{5-3}}$

$\therefore\ a^m \times a^n = a^{m+n}$

and $\dfrac{a^m}{a^n} = \begin{cases} a^{m-n}, & \text{when } m>n \\ \dfrac{1}{a^{n-m}}, & \text{when } n>m \end{cases}$

Formula 3. $(ab)^n = a^n \times b^n$

We observe, $(5 \times 2)^3 = (5 \times 2) \times (5 \times 2) \times (5 \times 2)$ $[\because a^3 = a \times a \times a; a = 5 \times 2]$

$= 5 \times 2 \times 5 \times 2 \times 5 \times 2$

$= (5 \times 5 \times 5) \times (2 \times 2 \times 2)$

$= 5^3 \times 2^3$

In general, $(ab)^n = ab \times ab \times ab \times .... \times ab$ [Successive multiplication of n times ab]

$$= (a \times a \times a \times ....\times a) \times (b \times b \times b \times ....\times b)$$

$$= a^n b^n$$

Formula 4. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, (b \neq 0)$

We observe, $\left(\frac{5}{2}\right)^3 = \frac{5}{2} \times \frac{5}{2} \times \frac{5}{2} = \frac{5 \times 5 \times 5}{2 \times 2 \times 2} = \frac{5^3}{2^3}$

In general, $\left(\frac{a}{b}\right)^n = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times ....\times \frac{a}{b}$ [Successive multiplication of n times $\frac{a}{b}$]

$$= \frac{a \times a \times a \times ......\times a}{b \times b \times b \times ......\times b} = \frac{a^n}{b^n}$$

Formula 5. $a^0 = 1, (a \neq 0)$

We have, $\frac{a^n}{a^n} = a^{n-n} = a^0$

Again, $\frac{a^n}{a^n} = \frac{a \times a \times a \times .....\times a}{a \times a \times a \times .....\times a}$ [in both the cases of num. and den multiplication of n times a]

$= 1$

$\therefore\ a^0 = 1.$

Formula 6. $a^{-n} = \frac{1}{a^n}, (a \neq 0)$

$a^{-n} = \frac{a^{-n} \times a^n}{1 \times a^n}$ [multiplying both num. and denom. by a^n]

$$= \frac{a^{-n+n}}{a^n} = \frac{a^0}{a^n} = \frac{1}{a^n}$$

$$\therefore\ a^{-n} = \frac{1}{a^n}$$

Rmark : $\frac{1}{a^n} = \frac{a^o}{a^n} = a^{o-n} = a^{-n}$

Formula 7. $(a^m)^n = a^{mn}$

$(a^m)^n = a^m \times a^m \times a^m \times ....\times a^m$ [successive multiplication of n times a^m]

$= a^{m+m+m+.........+m}$ [in the power, sum of n times of exponent m]

$$= a^{n\times m} = a^{mn}$$

$$\therefore\ (a^m)^n = a^{mn}$$

Example 1. Find the values (a) $\dfrac{5^2}{5^3}$ (b) $\left(\dfrac{2}{3}\right)^5 \times \left(\dfrac{2}{3}\right)^{-5}$

Solution : (a) $\dfrac{5^2}{5^3} = 5^{2-3} = 5^{-1} = \dfrac{1}{5^1} = \dfrac{1}{5}$

(b) $\left(\dfrac{2}{3}\right)^5 \times \left(\dfrac{2}{3}\right)^{-5} = \left(\dfrac{2}{3}\right)^{5-5} = \left(\dfrac{2}{3}\right)^0 = 1$

Example 2. Simplify :(a) $\dfrac{5^4 \times 8 \times 16}{2^5 \times 125}$ (b) $\dfrac{3\cdot 2^n - 4\cdot 2^{n-2}}{2^n - 2^{n-1}}$

Solution :(a) $\dfrac{5^4 \times 8 \times 16}{2^5 \times 125} = \dfrac{5^4 \times 2^3 \times 2^4}{2^5 \times 5^3} = \dfrac{5^4 \times 2^{3+4}}{5^3 \times 2^5} = \dfrac{5^4}{5^3} \times \dfrac{2^7}{2^5} = 5^{4-3} \times 2^{7-5}$

$= 5^1 \times 2^2 = 5 \times 4 = 20$

(b) $\dfrac{3\cdot 2^n - 4\cdot 2^{n-2}}{2^n - 2^{n-1}} = \dfrac{3\cdot 2^n - 2^2\cdot 2^{n-2}}{2^n - 2^n\cdot 2^{-1}} = \dfrac{3\cdot 2^n - 2^{2+n-2}}{2^n - 2^n\cdot \frac{1}{2}}$

$$= \frac{3\cdot 2^n - 2^n}{\left(1-\frac{1}{2}\right)\cdot 2^n} = \frac{(3-1)\cdot 2^n}{\frac{1}{2}\cdot 2^n} = \frac{2\cdot 2^n}{\frac{1}{2}\cdot 2^n} = 2\cdot 2 = 4.$$

Example 3. Show that $(a^p)^{q-r}\cdot(a^q)^{r-p}\cdot(a^r)^{p-q} = 1$

Solution : $(a^p)^{q-r}\cdot(a^q)^{r-p}\cdot(a^r)^{p-q}$

$= a^{p(q-r)}\cdot a^{q(r-p)}\cdot a^{r(p-q)}$ $[\because (a^m)^n = a^{mn}]$

$= a^{pq-pr}\cdot a^{qr-pq}\cdot a^{pr-qr}$

$= a^{pq-pr+qr-pq+pr-qr}$

$= a^0 = 1.$

Activity : Fill in the blank boxes :

(*i*) $3\times 3\times 3\times 3 = 3^{\square}$ (*ii*) $5^{\square}\times 5^3 = 5^5$ (*iii*) $a^2 \times a^{\square} = a^{-3}$ (*iv*) $\dfrac{4}{4^{\square}} = 1^{\square}$

(*v*) $(-5)^0 = \square$

4.3 n th root

We notice, $5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = \left(5^{\frac{1}{2}}\right)^2$

Again, $5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^{2\times\frac{1}{2}} = 5$

$\therefore \left(5^{\frac{1}{2}}\right)^2 = 5.$

Square (power 2) of $5^{\frac{1}{2}} = 5$ and square root (second root) of 5 = $5^{\frac{1}{2}}$

$5^{\frac{1}{2}}$ is written as $\sqrt{5}$ in terms of the sign $\sqrt{}$ of square root.

Again, we notice $5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}} = \left(5^{\frac{1}{3}}\right)^3$

Again, $5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}} = 5^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 5^{3\times\frac{1}{3}} = 5$

$\therefore \left(5^{\frac{1}{3}}\right)^3 = 5.$

Cube (power 3) of $5^{\frac{1}{3}} = 5$ and cube root (third root) of 5 = $5^{\frac{1}{3}}$.

$5^{\frac{1}{3}}$ is written as $\sqrt[3]{5}$ in terms of the sign $\sqrt[3]{}$ of cube root.

In the case of n th root,

$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots\ldots. \times a^{\frac{1}{n}}$ [successive multiplication of n times $a^{\frac{1}{n}}$]

$= \left(a^{\frac{1}{n}}\right)^n.$

Again, $a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots\ldots. \times a^{\frac{1}{n}}$

$= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\ldots\ldots\ldots+\frac{1}{n}}$ [in the exponent, sum of n times $\frac{1}{n}$]

$= a^{n\times\frac{1}{n}} = a$

$\therefore \left(a^{\frac{1}{n}}\right)^n = a.$

n th power of $a^{\frac{1}{n}} = a$ and n th root of $a = a^{\frac{1}{n}}$

i.e. n th power of $a^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^n = a$ and n th root of $a = (a)^{\frac{1}{n}} = a^{\frac{1}{n}} = \sqrt[n]{a}$. n th root of a is written as $\sqrt[n]{a}$.

Example 4. Simplify :(a) $7^{\frac{3}{4}} \cdot 7^{\frac{1}{2}}$ (b) $(16)^{\frac{3}{4}} \div (16)^{\frac{1}{2}}$ (c) $\left(10^{\frac{2}{3}}\right)^{\frac{3}{4}}$

Solution : (a) $7^{\frac{3}{4}} \cdot 7^{\frac{1}{2}} = 7^{\frac{3}{4}+\frac{1}{2}} = 7^{\frac{5}{4}}$

(b) $(16)^{\frac{3}{4}} \div (16)^{\frac{1}{2}} = \dfrac{(16)^{\frac{3}{4}}}{(16)^{\frac{1}{2}}} = (16)^{\frac{3}{4}-\frac{1}{2}} = (16)^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = (2)^{4\times\frac{1}{4}} = 2.$

(c) $\left(10^{\frac{2}{3}}\right)^{\frac{3}{4}} = 10^{\frac{2}{3}\times\frac{3}{4}} = 10^{\frac{1}{2}} = \sqrt{10}.$

Example 5. Simplify :(a) $(12)^{-\frac{1}{2}} \times \sqrt[3]{54}$ (b) $(-3)^3 \times \left(-\frac{1}{2}\right)^2$

Solution : (a) $(12)^{-\frac{1}{2}} \times \sqrt[3]{54} = \dfrac{1}{(12)^{\frac{1}{2}}} \times (54)^{\frac{1}{3}}$

$$= \frac{1}{(2^2 \times 3)^{\frac{1}{2}}} \times (3^3 \times 2)^{\frac{1}{3}} = \frac{1}{(2^2)^{\frac{1}{2}} \cdot \times 3^{\frac{1}{2}}} \times (3^3)^{\frac{1}{3}} \cdot 2^{\frac{1}{3}}$$

$$= \frac{1}{(2\cdot 3)^{\frac{1}{2}}} \times (3\cdot 2)^{\frac{1}{3}} = \frac{2^{\frac{1}{3}}}{2^1} \times \frac{3^1}{3^{\frac{1}{2}}} = \frac{3^{1-\frac{1}{2}}}{2^{1-\frac{1}{3}}} = \frac{3^{\frac{1}{2}}}{2^{\frac{2}{3}}} = \frac{3^{\frac{1}{2}}}{4^{\frac{1}{3}}} = \frac{\sqrt{3}}{\sqrt[3]{4}}.$$

(b) $(-3)^3 \times \left(-\frac{1}{2}\right)^2 = (-3)(-3)(-3) \times \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)$

$= -27 \times \frac{1}{4} = -\frac{27}{4}$

Activity : Simplify : (*i*) $\dfrac{2^4 \cdot 2^2}{32}$ (*ii*) $\left(\frac{2}{3}\right)^{\frac{5}{2}} \times \left(\frac{2}{3}\right)^{\frac{-5}{2}}$ (*iii*) $8^{\frac{3}{4}} \div 8^{\frac{1}{2}}$

To be noticed :

1. Under the condition $a > 0, a \neq 1$, if $a^x = a^y$, $x = y$
2. Under the condition $a > 0, b > 0, x \neq 0$, if $a^x = b^x$, $a = b$

Example 6. Solve : $4^{x+1} = 32$.

Solution : $4^{x+1} = 32$

or $(2^2)^{x+1} = 32$, or, $2^{2x+2} = 2^5$ [if $a^x = a^y$, $x = y$]

$\therefore 2x + 2 = 5,$

or, $2x = 5 - 2$, or, $2x = 3$

$\therefore x = \dfrac{3}{2}$

$\therefore$ Solution is $x = \dfrac{3}{2}$

Exercise 4.1

Simplify (1 – 10) :

1. $\dfrac{3^3 \cdot 3^5}{3^6}$ 2. $\dfrac{5^3 \cdot 8}{2^4 \cdot 125}$ 3. $\dfrac{7^3 \times 7^{-3}}{3 \times 3^{-4}}$ 4. $\dfrac{\sqrt[3]{7^2} \cdot \sqrt[3]{7}}{\sqrt{7}}$

5. $(2^{-1} + 5^{-1})^{-1}$ 6. $(2a^{-1} + 3b^{-1})^{-1}$ 7. $\left(\dfrac{a^2 b^{-1}}{a^{-2} 6}\right)^2$

8. $\sqrt{x^{-1}y} \cdot \sqrt{y^{-1}z} \cdot \sqrt{z^{-1}x}, (x > 0, y > 0, z > 0)$ 9. $\dfrac{2^{n+4} - 4 \cdot 2^{n+1}}{2^{n+2} \div 2}$ 10. $\dfrac{3^{m+1}}{(2^m)^{m-1}} \div \dfrac{3^{m+1}}{(3^{m-1})^{m+1}}$

Prove (11 – 18) :

11. $\dfrac{4^n - 1}{2^n - 1} = 2^n + 1$

12. $\dfrac{2^{p+1} \cdot 3^{2p-q} \cdot 5^{p+q} \cdot 6^p}{6^q \cdot 10^{p+2} \cdot 15^q} = \dfrac{1}{50}$

13. $\left(\dfrac{a^\ell}{a^m}\right)^n \cdot \left(\dfrac{a^m}{a^n}\right)^\ell \cdot \left(\dfrac{a^n}{a^\ell}\right)^m = 1$

14. $\dfrac{a^{p+q}}{a^{2r}} \times \dfrac{a^{q+r}}{a^{2p}} \times \dfrac{a^{r+p}}{a^{2q}} = 1$

15. $\left(\dfrac{x^a}{x^b}\right)^{\frac{1}{ab}} \cdot \left(\dfrac{x^b}{x^c}\right)^{\frac{1}{bc}} \cdot \left(\dfrac{x^c}{x^a}\right)^{\frac{1}{ca}} = 1$

16. $\left(\dfrac{x^a}{x^b}\right)^{a+b} \cdot \left(\dfrac{x^b}{x^c}\right)^{b+c} \cdot \left(\dfrac{x^c}{x^a}\right)^{c+a} = 1$

17. $\left(\dfrac{x^p}{x^q}\right)^{p+q-r} \times \left(\dfrac{x^q}{x^r}\right)^{q+r-p} \times \left(\dfrac{x^r}{x^p}\right)^{r+p-q} = 1$

18. If $a^x = b$, $b^y = c$ and $c^z = a$, show that $xyz = 1$

Solve (19 – 22) :

19. $4^x = 8$ 20. $2^{2x+1} = 128$ 21. $\left(\sqrt{3}\right)^{x+1} = \left(\sqrt[3]{3}\right)^{2x-1}$ 22. $2^x + 2^{1-x} = 3$

4.4 Logarithm

Logarithm is used to find the values of exponential expressions. Logarithm is written in brief as ‘*Log*’. Product, quotient, etc. of large numbers or quantities can easily be determined by the help of log.

We know, $2^3 = 8$; this mathematical statement is written in terms of log as $\log_2 8 = 3$. Again, conversely, if $\log_2 8 = 3$, it can be written in terms of exponents as $2^3 = 8$. That is, if $2^3 = 8$, then $\log_2 8 = 3$ and conversely, if $\log_2 8 = 3$, then $2^3 = 8$. Similarly, $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ can be written in terms of log as $\log_2 \frac{1}{8} = -3$.

If $a^x = N$, ($a > 0, a \neq 1$), $x = \log_a N$ is defined as a based $\log N$.

To be noticed : Whatever may be the values of x, positive or negative, a^x is always positive. So, only the log of positive numbers has values which are real ; log of zero or negative numbers have no real value.

Activity-1 : Express in terms of log :	**Activity-2 :** Fill in the blanks :	
(i) $10^2 = 100$	in terms of exponent	in terms of log
(ii) $3^{-2} == \frac{1}{9}$	$10^0 = 1$	$\log_{10} 1 = 0$
(iii) $2^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}$	$e^0 = ...$	$\log_e 1 = ...$
(iv) $\sqrt{2^4}$	$a^0 = 1$	$...... = ...$
	$10^1 = 10$	$\log_{10} 10 = 1$
	$e^1 = ...$	$....... = ...$
	$....... = ...$	$\log_a a = 1$

Formulae of Logarithms :

Let, $a > 0, a \neq 1; b > 0, b \neq 1$ and $M > 0, N > 0$.

Formula 1. (a) $\log_a 1 = 0, (a > 0, a \neq 1)$

(b) $\log_a a = 1, (a > 0, a \neq 1)$

Proof : (a) We know from the formula of exponents, $a^0 = 1$

$\therefore$ from the definition of log, we get, $\log_a 1 = 0$ (proved)

(b) We know, from the formula of exponents, $a^1 = a$

$\therefore$ from the definition of log, we get, $\log_a a = 1$ (proved).

Formula 2. $\log_a(MN) = \log_a M + \log_a N$

Proof : Let, $\log_a M = x, \log_a N = y$;

$\therefore \; M = a^x, N = a^y$

Now, $MN = a^x \cdot a^y = a^{x+y}$

$\therefore\ \log_a(MN) = x + y,$ or $\log_a(MN) = \log_a M + \log_a N$ [putting the values of x, y]

$\therefore\ \log_a(MN) = \log_a M + \log_a N.$ (proved)

Note 1. $\log_a(MNP.....) = \log_a M + \log_a N + \log_a P + ...$

Note 2. $\log_a(M \pm N) \neq \log_a M \pm \log_a N$

Formula 3. $\log_a \frac{M}{N} = \log_a M - \log_a N$

Proof : Let $\log_a M = x, \log_a N = y$;

$\therefore\ M = a^x, N = a^y$

Now, $\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y}$

$\therefore\ \log_a\left(\frac{M}{N}\right) = x - y$

$\therefore\ \log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$ (proved).

Formula 4. $\log_a M^r = r \log_a M.$

Proof : :Let $\log_a M = x;\ \therefore M = a^x$

$\therefore\ (M)^r = (a^x)^r;$ or $M^r = a^{rx}$

$\therefore\ \log_a M^r = rx;$ or $\log_a M^r = r \log_a M$

$\therefore\ \log_a M^r = r \log_a M.$ (proved).

N.B. : $(\log_a M)^r \neq r \log_a M$

Formula 5. $\log_a$ M = $\log_b$ M × $\log_a$ b, (change of base)

Proof : Let, $\log_a M = x, \log_b M = y$

$\therefore\ a^x = M, b^y = M$

$\therefore\ a^x = b^y,$ or $(a^x)^{\frac{1}{y}} = (b^y)^{\frac{1}{y}}$

or $b = a^{\frac{x}{y}}$

$\therefore\ \frac{x}{y} = \log_a b,$ or $x = y \log_a b$

or , $x = y \log_a b,$ or $\log_a M = \log_b M \times \log_a b$ (proved).

Corollary : $\log_a b = \frac{1}{\log_b a},$ or, $\log_b a = \frac{1}{\log_a b}$

Proof : We know, $\log_a M = \log_b M \times \log_a b$ [formula 5]

Putting $M = a$ we get,

$\log_a a = \log_b a \times \log_a b$

or $1 = \log_b a \times \log_b b$;

$\therefore \log_b a = \dfrac{1}{\log_a b}$

or $\log_a b = \dfrac{1}{\log_b a}$ (proved).

Example 7. Find the value : (a) $\log_{10} 100$ (b) $\log_3\left(\dfrac{1}{9}\right)$ (c) $\log_{\sqrt{3}} 81$

Solution:

(a) $\log_{10} 100 = \log_{10} 10^2 = 2\log_{10} 10$ $[\because \log_{10} M^r = r\log_{10} M]$

$= 2 \times 1$ $[\because \log_a a = 1]$ $= 2$

(b) $\log_3\left(\dfrac{1}{9}\right) = \log_3\left(\dfrac{1}{3^2}\right) = \log_3 3^{-2} = -2\log_3 3$ $[\because \log_a M^r = r\log_a M]$

$= -2 \times 1[\because \log_a a = 1] = -2$

(c) $\log_{\sqrt{3}} 81 = \log_{\sqrt{3}} 3^4 = \log_{\sqrt{3}}\{(\sqrt{3})^2\}^4 = \log_{\sqrt{3}}\left(\sqrt{3}\right)^8$

$= 8\log_{\sqrt{3}}\sqrt{3}[\because \log_a M^r = r\log_a M]$

$= 8 \times 1, [\because \log_a a = 1]$

$= 8$

Example 8. (a) What is the log of $5\sqrt{5}$ to the base 5 ?

(b) $\log 400 = 4$; what is the base ?

Solution : (a) $5\sqrt{5}$ to the base 5

$= \log_5 5\sqrt{5} = \log_5(5 \times 5^{\frac{1}{2}}) = \log_5 5^{\frac{3}{2}}$

$= \dfrac{3}{2}\log_5 5, [\because \log_a M^r = r\log_a M]$

$= \dfrac{3}{2} \times 1, [\because \log_a a = 1]$

$= \dfrac{3}{2}$

(b) Let the base be a.

$\therefore$ by the question, $\log_a 400 = 4$

$\therefore$ $a^4 = 400$

or $a^4 = (20)^2 = \{(2\sqrt{5})^2\}^2 = (2\sqrt{5})^4$

or $a^4 = (2\sqrt{5})^4$

$\therefore\ a = 2\sqrt{5}$ $\quad [\because$ if $a^x = b^x$, $a = b]$

$\therefore$ the base is $2\sqrt{5}$

Example 9. Find the value of x :

(a) $\log_{10} x = -2$ (b) $\log_x 324 = 4$

Solution :

(a) $\log_{10} x = -2$

$\therefore x = 10^{-2}$

$\therefore\ x = \dfrac{1}{10^2} = \dfrac{1}{100} = 0 \cdot 01$

$\therefore\ x = 0 \cdot 01$

(b) $\log_x 324 = 4$

$\therefore x^4 = 324 = 3 \times 3 \times 3 \times 3 \times 2 \times 2$

$= 3^4 \times 2^2 = 3^4 \times (\sqrt{2})^4$

or $x^4 = (3\sqrt{2})^4$

$\therefore x = 3\sqrt{2}.$

Example 10. Prove that, $3\log_{10} 2 + \log_{10} 5 = \log_{10} 40$

Solution :Left hand side = $3\log_{10} 2 + \log_{10} 5$

$= \log_{10} 2^3 + \log_{10} 5, [\because \log_a M^r = r\log_a M]$

$= \log_{10} 8 + \log_{10} 5$

$= \log_{10}(8 \times 5), [\because \log_a(MN) = \log_a M + \log_a N]$

$= \log_{10} 40$ = Right hand side (proved).

Example 11. Simplify : $\dfrac{\log_{10}\sqrt{27} + \log_{10} 8 - \log_{10}\sqrt{1000}}{\log_{10} 1 \cdot 2}$

Solution : $\dfrac{\log_{10}\sqrt{27} + \log_{10} 8 - \log_{10}\sqrt{1000}}{\log_{10} 1 \cdot 2}$

$$= \frac{\log_{10}(3^3)^{\frac{1}{2}} + \log_{10} 2^3 - \log_{10}(10^3)^{\frac{1}{2}}}{\log_{10}\frac{12}{10}}$$

$$= \frac{\log_{10} 3^{\frac{3}{2}} + \log_{10} 2^3 - \log_{10} 10^{\frac{3}{2}}}{\log_{10} 12 - \log_{10} 10}$$

$$= \frac{\frac{3}{2}\log_{10} 3 + 3\log_{10} 2 - \frac{3}{2}\log_{10} 10}{\log_{10}(3 \times 2^2) - \log_{10} 10}$$

$$= \frac{\frac{3}{2}(\log_{10} 3 + 2\log_{10} 2 - 1)}{(\log_{10} 3 + 2\log_{10} 2 - 1)} \qquad [\because \log_{10} 10 = 1]$$

$$= \frac{3}{2}.$$

Exercise 4.2

1. Find the value :(a) $\log_3 81$ (b) $\log_5 \sqrt[3]{5}$ (c) $\log_4 2$ (d) $\log_{2\sqrt{5}} 400$
   (e) $\log_5 \left(\sqrt[3]{5} \cdot \sqrt{5}\right)$
2. Find the value of x :(a) $\log_5 x = 3$ (b) $\log_x 25 = 2$ (c) $\log_x \frac{1}{16} = -2$
3. Show that,
   (a) $5\log_{10} 5 - \log_{10} 25 = \log_{10} 125$
   (b) $\log_{10} \frac{50}{147} = \log_{10} 2 + 2\log_{10} 5 - \log_{10} 3 - 2\log_{10} 7$
   (c) $3\log_{10} 2 + 2\log_{10} 3 + \log_{10} 5 = \log_{10} 360$
4. Simplify :
   (a) $7\log_{10} \frac{10}{9} - 2\log_{10} \frac{25}{24} + 3\log_{10} \frac{81}{80}$
   (b) $\log_7 \left(\sqrt[5]{7} \cdot \sqrt{7}\right) - \log_3 \sqrt[3]{3} + \log_4{}^{2}$
   (c) $\log_e \frac{a^3 b^3}{c^3} + \log_e \frac{b^3 c^3}{d^3} + \log_e \frac{c^3 d^3}{a^3} - 3\log_e b^2 c$

4.5 Scientific or Standard form of numbers

We can express very large numbers or very small numbers in easy and small form by exponents.

Such as, velocity of light $= 300000$ km/sec $= 300000000$ m/sec
$= 3 \times 100000000$ m/sec. $= 3 \times 10^8$ m/sec.

Again, radius of a hydrogen atom $= 0 \cdot 00000000037$ cm

$$= \frac{37}{10000000000} \text{ cm} = 37 \times 10^{-10} \text{ cm}$$

$$= 3 \cdot 7 \times 10 \times 10^{-10} \text{ cm} = 3 \cdot 7 \times 10^{-9} \text{ cm}$$

For convenience, very large number or very small number is expressed in the form $a \times 10^n$, where $1 \le a < 10$ and $n \in Z$. The form $a \times 10^n$ of any number is called the scientific or standard form of the number.

Activity : Express the following numbers in scientific form :
(a) 15000 (b) $0 \cdot 000512$

4.6 Logarithmic Systems

Logarithmic systems are of two kinds :

(a) Natural Logarithm :

The mathematician John Napier (1550 –1617) of Scotland first published the book on logarithm in 1614 by taking e as its base. e is an irrational number, $e = 2 \cdot 718.......$ Such logarithm is called Napierian logarithm or e based logarithm or natural logarithm. $\log_e x$ is also written in the form $\ln x$.

(b) Common Logarithm :

The mathematician Henry Briggs (1561 – 1630) of England prepared log table in 1624 by taking 10 as the base. Such logarithm is called Briggs logarithm or 10 based logarithm or practical logarithm.

N.B. : If there is no mention of base, e in the case of expression (algebraic) and 10, in the case of number are considered the base. In log table 10 is taken as the base.

4.7 Characteristic and Mantissa of Common Logarithm

(a) Characteristics :

Let a number N be expressed in scientific form as $N = a \times 10^n$, where $N > 0, 1 \le a < 10$ and $n \in Z$.

Taking log of both sides with base 10,

$$\log_{10} N = \log_{10}(a \times 10^n)$$

$$= \log_{10} a + \log_{10} 10^n = \log_{10} a + n \log_{10} 10$$

$$\therefore \log_{10} N = n + \log_{10} a \quad [\because \log_{10} 10 = 1]$$

$$\therefore \log_{10} N = n + \log_{10} a$$

Suppressing the base 10, we have,

$$\log N = n + \log a$$

n is called the characteristic of $\log N$.

We observe : Table-1

N	Form $a \times 10^m$ of N	Exponent	Number of digits on the left of the decimal point	Characteristic
6237	$6 \cdot 237 \times 10^3$	3	4	$4 - 1 = 3$
$623 \cdot 7$	$6 \cdot 237 \times 10^2$	2	3	$3 - 1 = 2$
$62 \cdot 37$	$6 \cdot 237 \times 10^1$	1	2	$2 - 1 = 1$
$6 \cdot 237$	$6 \cdot 237 \times 10^0$	0	1	$1 - 1 = 0$
$0 \cdot 6237$	$6 \cdot 237 \times 10^{-1}$	-1	0	$0 - 1 = -1$

We observe : Table-2

N	Form $a\times10^{m}$ of N	Exponent	Number of zeroes between decimal point and its next first significant digit	Characteristic
$0\cdot6237$	$6\cdot237\times10^{-1}$	-1	0	$-(0+1)=-1$
$0\cdot06237$	$6\cdot237\times10^{-2}$	-2	1	$-(1+1)=-2$
$0\cdot006237$	$6\cdot237\times10^{-3}$	-3	2	$-(2+1)=-3$

We observe from table-1 :

As many digits are there in the integral part of a number, characteristic of log of the number will be 1 less than that number of digits and that will be positive.

We observe from table 2 :

If there is no integral part of a number, as many zeroes are there in between decimal point and its next first significant digit, the characteristic of log of the number will be 1 more than the number of zeroes and that will be negative.

N. B. 1. Characteristic may be either positive or negative, but the mantissa will always be positive.

N. B. 2. If any characteristic is negative, not placing '–' sign on the left of the characteristic, it is written by giving '–' (bar sign) over the characteristic. Such as, characteristic –3 will be written as $\bar{3}$. Otherwise, whole part of the log including mantissa will mean negative.

Example 12. Find the characteristics of log of the following numbers :

(i) 5570 (ii) $45\cdot70$ (iii) $0\cdot4305$ (iv) $0\cdot000435$

Solution : (i) $5570=5\cdot570\times1000=5\cdot570\times10^{3}$

$\therefore$ Characteristic of log of the number is 3.

Otherwise, number of digits in the number 5570 is 4.

$\therefore$ Characteristic of log of the number is $=4-1=3$

$\therefore$ Characteristic of log of the number is 3.

(ii) $45\cdot70=4\cdot570\times10^{1}$

$\therefore$ Characteristic of log of the number is 1.

Otherwise, there are 2 digits in the integral part (i.e. on left of decimal point) of the number.

$\therefore$ Characteristic of the log of the number is $=2-1=1$

$\therefore$ Characteristic of log of the number is 45 ·70 is 1.

(iii) $0\cdot4305=4\cdot305\times10^{-1}$

$\therefore$ Characteristic of log of the number is -1

Otherwise, there is no significant digit in the integral part (before the decimal point) of the number or there is zero digit.

$\therefore$ Characteristic of log of the number $=0-1=-1=\bar{1}$

Again, there is no zero in between decimal point and its next first significant digit of the number $0 \cdot 4305$, i.e. there is 0 zeroes.

$\therefore$ Characteristic of log of the number is $= -(0+1) = -1 = \bar{1}$

$\therefore$ Characteristic of log of the number 0 ·4305 is $\bar{1}$

(*iv*) $0 \cdot 000435 = 4 \cdot 35 \times 10^{-4}$

$\therefore$ Characteristic of log of the number is -4 or $\bar{4}$

Otherwise, there are 3 zeroes in between decimal point and its next 1st significant digit.

$\therefore$ Characteristic of log of the number is $= -(3+1) = -4 = \bar{4}$

$\therefore$ Characteristic of log of the number is $0 \cdot 000435$ is $\bar{4}$

(b) Mantissa :

Mantissa of the Common Logarithm of any number is a non-negative number less than 1. It is mainly an irrational number. But the value of mantissa can be determined upto a certain places of decimal.

Mantissa of the log of a number can be found from log table. Again, it can also be found by calculator. We shall find the mantissa of the log of any number in 2nd method, that is by calculator.

Determination of common logarithm with the help of calculator :

Example 13. Find the characteristic and mantissa of $\log 2717$:

Solution :We use the calculator :

$\boxed{AC}$ $\boxed{\log}$ $\boxed{2717}$ $\boxed{=}$ $3 \cdot 43408$

$\therefore$ Characteristic of $\log 2717$ is 3 and mantissa is $\cdot 43408$

Example 14. Find the characteristic and mantissa of $\log\ 43 \cdot 517$.

Solution :We use the calculator :

$\boxed{AC}$ $\boxed{\log}$ $\boxed{43{\cdot}517}$ $\boxed{=}$ 1·63866

$\therefore$ Characteristic of log 43 ·517 is 1 and mantissa is ·63866

Example 15. What are the characteristic and mantissa of the log of $0 \cdot 00836$?

Solution :We use the calculator :

$\boxed{AC}$ $\boxed{\log}$ $\boxed{0{\cdot}00836}$ $\boxed{=}$ -3 ·92221 $= \bar{3}{\cdot}92221$

$\therefore$ Characteristic of log 0 ·00836 is -3 or $\bar{3}$ and mantissa is ·92221

Example 16. Find $\log_e 10$

Solution : $\log_e 10 = \dfrac{1}{\log_{10} e} = \dfrac{1}{\log_{10}\ 2 \cdot 71828}$ [taking the value of e upto five decimal places]

$= \dfrac{1}{0 \cdot 43429}$ [using calculator]

= 2·30259 (approx).

Alternative : We use the calculator :

$\boxed{AC}$ $\boxed{\ln}$ $\boxed{10}$ $\boxed{=}$ 2·30259 (approx).

> **Activity :** Find the logarithm of the following numbers (each with the base 10 and e) by using calculator : (i) 2550 (ii) 52 ·143 (iii) 0·4145 (iv) 0·0742

Exercise 4.3

1. On what condition $a^0 = 1$?

   a. $a = 0$ b. $a \neq 0$ c. $a > 0$ d. $a \neq 1$

2. Which one of the following is the value of $\sqrt[3]{5} \cdot \sqrt[3]{5}$?

   a. $\sqrt[6]{5}$ b. $\left(\sqrt[3]{5}\right)^3$ c. $\left(\sqrt{5}\right)^3$ d. $\sqrt[3]{25}$

3. On what exact condition $\log_a a = 1$?

   a. $a > 0$ b. $a \neq 1$ c. $a > 0, a \neq 1$ d. $a \neq 0, a > 1$

4. If $\log_x 4 = 2$, what is the value of x ?

   a. 2 b. ± 2 c. 4 d. 10

5. What is the condition for which a number is to be written in the form $a \times 10^n$?

   a. $1 < a < 10$ b. $1 \leq a \leq 10$ c. $1 \leq a < 10$ d. $1 < a \leq 10$

6. Observe the following information :

   i. $\log_a (m)^p = p\log_a m$

   ii. $2^4 = 16$ and $log_2 16 = 4$ are synonymous.

   iii. $\log_a (m+n) = \log_a m + \log_a n$

Which of the above information are correct ?

   a. i and ii b. ii and iii c. i and iii d. i, ii and iii

7. What is the characteristic of the common log of $0 \cdot 0035$?

   a. 3 b. 1 c. $\overline{2}$ d. $\overline{3}$

8. Cnsidering the number $0 \cdot 0225$, answer the following questions :

   (1) Which one of the following is of the form a^n of the number ?

   a. $(2 \cdot 5)^2$ b. $(\cdot 015)^2$ c. $(1 \cdot 5)^2$ d. $(\cdot 15)^2$

   (2) Which one of the following is the scientific form of the number ?

   a. 225×10^{-4} b. $22 \cdot 5 \times 10^{-3}$ c. $2 \cdot 25 \times 10^{-2}$ d. $\cdot 225 \times 10^{-1}$

   (3) What is the characteristic of the common log of the number ?

   a. $\overline{2}$ b. $\overline{1}$ c. 0 d. 2

9. Express into scientific form :
   (a) 6530 (b) $60 \cdot 831$ (c) $0 \cdot 000245$ (d) 37500000
   (e) $0 \cdot 00000014$
10. Express in the form of ordinary decimals :
   (a) 10^5 (b) 10^{-5} (c) $2 \cdot 53 \times 10^4$ (d) $9 \cdot 813 \times 10^{-3}$
   (e) $3 \cdot 12 \times 10^{-5}$
11. Find the characteristic of common logarithm of the following numbers (without using calculator) :
   (a) 4820 (b) $72 \cdot 245$ (c) $1 \cdot 734$ (d) $0 \cdot 045$
   (e) $0 \cdot 000036$
12. Find the characteristic and mantissa of the common logarithm of the following numbers by using calculator :
   (a) 27 (b) $63 \cdot 147$ (c) $1 \cdot 405$ (d) $0 \cdot 0456$
   (e) $0 \cdot 000673$
13. Find the common logarithm of the product/quotient (approximate value upto five decimal places) :
   (a) $5 \cdot 34 \times 8 \cdot 7$ (b) $0 \cdot 79 \times 0 \cdot 56$ (c) $22 \cdot 2642 \div 3 \cdot 42$
   (d) $0 \cdot 19926 \div 32 \cdot 4$
14. If $\log 2 = 0 \cdot 30103, \log 3 = 0 \cdot 47712$ and $\log 7 = 0 \cdot 84510$, find the value of the following expressions :
   (a) $\log 9$ (b) $\log 28$ (c) $\log 42$
15. Given, $x = 1000$ and $y = 0 \cdot 0625$

   a. Express x in the form $a^n b^n$, where a and b are prime numbers.

   b. Express the product of x and y in scientific form.

   c. Find the characteristic and mantissa of the common logarithm of xy.